# A Central Limit Theorem for Integer Partitions into Small Powers

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# Plan for the Following 25 Minutes

- **Integer partitions** $42 = 1 + 2 + 6 + 10 + 23$
  - restricted and unrestricted partitions
  - classical results
- **Variants** $42 = 2^2 + 2^2 + 3^2 + 5^2$
  - partitions into powers
  - primes as summands
- **Partitions into small powers** $42 = \lfloor\sqrt{7}\rfloor + \lfloor\sqrt{379}\rfloor + \lfloor\sqrt{449}\rfloor$
  - our result $\rightsquigarrow$ central limit theorem
  - idea of the proof

$\rightsquigarrow$ **enumerative** and **analytic combinatorics**

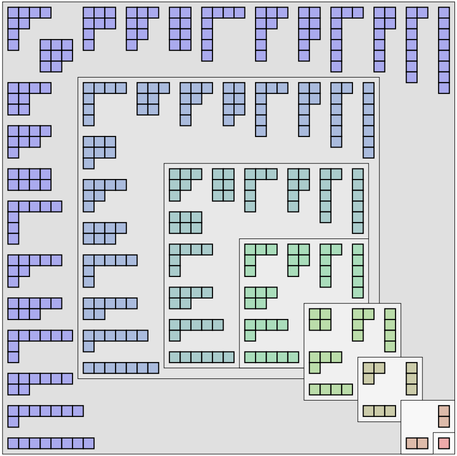
A **partition** of $n \in \mathbb{N}$ is a
**decomposition** of $n$ as a **sum of positive integers**,
**disregarding the order** of the summands.

$$
\begin{aligned}
42 &= 42 && (1)\\
&= 23 + 11 + 4 + 4 && (2)\\
&= 30 + 10 + 2 && (3)\\
&= 22 + 20 = 20 + 22 && (4)\\
&\;\vdots
\end{aligned}
$$

$$
\begin{aligned}
4 &= 4 && (1)\\
&= 3 + 1 && (2)\\
&= 2 + 2 && (3)\\
&= 2 + 1 + 1 && (4)\\
&= 1 + 1 + 1 + 1 && (5)
\end{aligned}
$$

$$p(n) = \#\text{ different partitions of } n \qquad \Rightarrow p(4) = 5$$

# Integer Partitions – Ferrer Diagrams



“Ferrer partitioning diagrams showing the paritions of positive integers 1 through 8”
created by R. A. Nonenmacher and shared via Wikimedia Commons under CC BY-SA 4.0

A partition is called **restricted**
if all summands are **distinct**.

$$
\begin{aligned}
42 &= 42 && (1)\\
&= \cancel{23 + 11 + 4 + 4} && \\
&= 30 + 10 + 2 && (2)\\
&= 22 + 20 = 20 + 22 && (3)\\
&\;\;\vdots
\end{aligned}
$$

$$
\begin{aligned}
4 &= 4 && (1)\\
&= 3 + 1 && (2)\\
&= \cancel{2 + 2} && \\
&= \cancel{2 + 1 + 1} && \\
&= \cancel{1 + 1 + 1 + 1} &&
\end{aligned}
$$

$q(n) = \#$ different **restricted** partitions of $n$ $\quad \Rightarrow q(4) = 2$

# Classical Results for the Partition Function

$$P(z) = 1 + \sum_{n\geq 1} p(n) z^n = \prod_{k\geq 1} \frac{1}{1-z^k}$$

$$Q(z) = 1 + \sum_{n\geq 1} q(n) z^n = \prod_{k\geq 1} (1+z^k)$$

**Hardy–Ramanujan (1918), Uspensky (1920)**, Erdős (1942):

$$p(n) \sim \frac{1}{4\sqrt{3}n} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

**Rademacher (1937)** provided an asymptotic expansion for $p(n)$.

Corollary of a theorem by **Meinardus (1954)**:

$$q(n) \sim \frac{1}{4 \cdot 3^{1/4} n^{3/4}} \exp\left(\pi\sqrt{\frac{n}{3}}\right)$$

Summands have to be

- primes $n = p_1 + \cdots + p_\ell$
- powers with fixed exponent $k \in \mathbb{N}$ $n = a_1^k + \cdots + a_\ell^k$
- prime powers with fixed exponent $k \in \mathbb{N}$ $n = p_1^k + \cdots + p_\ell^k$
- integers in sets with certain conditions
- integers with digital restrictions
- **small powers** with fixed exponent $\alpha \in \mathbb{Q}$ or $\alpha \in \mathbb{R}$ and $0 < \alpha < 1$

$$n = \lfloor a_1^\alpha \rfloor + \cdots + \lfloor a_\ell^\alpha \rfloor$$

$\hookrightarrow$ unrestricted and restricted (**$\mathbf{a_i}$ disctinct**, not the summands!)

$\hookrightarrow$ circle method and saddle point method

# Partitions into Small Powers

For $\alpha \in \mathbb{R}$ with $0 < \alpha < 1$, we consider **restricted partitions**

$$n = \lfloor a_1^\alpha \rfloor + \cdots + \lfloor a_\ell^\alpha \rfloor$$

with $1 \leq a_1 < \cdots < a_\ell$. Let $\omega_n$ be the random variable counting the **number of summands in a random partition** of the above form.

## Central Limit Theorem (L–Madritsch–Tichy 2022)

The random variable $\omega_n$ is **asymptotically normally dristributed**, i.e.,

$$\mathbb{P}\left(\frac{\omega_n - \mu_n}{\sigma_n} < x\right) = \frac{1}{2\pi} \int_{-\infty}^{x} e^{-t^2/2}\,\mathrm{d}t + o(1)$$

with mean $\mu_n$ and variance $\sigma_n^2$ satisfying

$$\mu_n \sim c_1 n^{1/(\alpha+1)} \quad \text{and} \quad \sigma_n \sim c_2 n^{1/(\alpha+1)},$$

where $c_1$ and $c_2$ are explicitly known.

# Overview of the Proof

**Analytic parts:**

- Mellin transform
- saddle-point method

**Probabilistic part:**

- Curtiss' theorem for moment-generating functions

By Curtiss' theorem, it is enough to show that

$$M_n(t) = \mathbb{E}\left(e^{(\omega_n-\mu_n)t/\sigma_n}\right) = e^{-\mu_n/\sigma_n}\mathbb{E}\left(e^{\omega_n t/\sigma_n}\right) \xrightarrow{n\to\infty} e^{t^2/2}.$$

**Generating function**, where $u$ counts the length of the partition:

$$Q(z,u) = 1 + \sum_{n\geq 1}\sum_{k\geq 1} q(n,k)u^k z^n = \prod_{k\geq 1}(1+uz^k)^{g(k)}$$

where $g(k)$ is given by

$$g(k) = \lceil (k+1)^{1/\alpha}\rceil - \lceil k^{1/\alpha}\rceil.$$

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### Lemma

For the expected value $\mathbb{E}(\omega_n) = \mu_n$ and the variance $\mathbb{V}(\omega_n) = \sigma_n^2$ we have

$$\mu_n = \frac{[z^n]Q_u(z,1)}{[z^n]Q(z,1)} \quad \text{and} \quad \sigma_n^2 = \frac{[z^n]Q_{uu}(z,1)}{[z^n]Q(z,1)} + \frac{[z^n]Q_u(z,1)}{[z^n]Q(z,1)} - \mu_n^2.$$

**Determine the coefficients** of $Q(z,u)$:

$$[z^n]Q(z,u) = \frac{1}{2\pi i}\oint_{|z|=e^{-r}} z^{-n-1}Q(z,u)\,\mathrm{d}z = \frac{e^{nr}}{2\pi}\int_{-\pi}^{\pi}\exp\big(\underbrace{int + f(r+it,u)}_{=:g(r+it)}\big)\,\mathrm{d}t$$

with suitable $r > 0$ and

$$f(\tau,u) = \log Q(e^{-\tau},u) = \sum_{k\geq 1} g(k)\log\big(1+ue^{-k\tau}\big)$$

Split the integral at $t_n = r^{1+3/(7\alpha)}$ and use **Taylor expansion** of $g(r+it)$:

$$\int_{|t|<t_n} e^{-\frac{t^2}{2}g''(r)}\left(1+O\left(\sup_t\left|t^3 g'''(r+it)\right|\right)\right)\mathrm{d}t$$

$\hookrightarrow$ analyse $g''$ and $g'''$ (**Mellin transform**) and estimate the error

# Recap

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Thank you for your attention!