Constructions of Large Progression-Free Sets, Caps and Related Structures

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DOCTORAL PROGRAM
DISCRETE MATHEMATICS

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Plan for the Following 45 Minutes

- **Progression-free sets in various settings**
  - in the integers (classical results)
  - in the affine space $\mathbb{Z}_m^n$
- **Caps**
  - in the affine space
  - in the projective space
- Connection to linear codes

Progression-Free Sets in the Integers (I)

$r_k(S)$ ... size of the largest k-term arithmetic progression-free subset of a set S

Some Exact Values for $S = \{1, \ldots, N\}$

- $r_3(\{1, 2, 3\}) = 2$
- $r_3(\{1, 2, 3, 4\}) = 3$
- $r_3(\{1, 2, 3, 4, 5\}) = 4$
- $r_3(\{1, 2, 3, 4, 5, 6\}) = 4$
- $r_3(\{1, 2, 3, 4, 5, 6, 7\}) = 4$
- $r_3(\{1, 2, 3, 4, 5, 6, 7, 8\}) = 4$

Salem and Spencer (1942):

$$r_3(\{1,\dots,N\}) > \frac{N}{\exp\left((\log 2+\varepsilon)\frac{\log N}{\log\log N}\right)}, \qquad N \geq N_\varepsilon$$

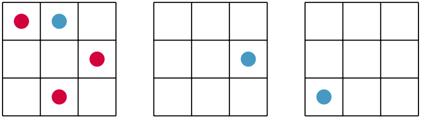
- integers in $(2d-1)$-ary digit system $\quad\rightsquigarrow k = \sum_{i\geq 0} a_i(2d-1)^i$
- using digits $0 \leq a_i \leq d-1$
- each a_i with frequency n/d for integers $\leq N = (2d-1)^n$
- no wrap mod $2d-1$

Behrend (1946):

$$r_3(\{1,\dots,N\}) > \frac{N}{\exp\left((2\sqrt{2\log 2}+\varepsilon)\sqrt{\log N}\right)}, \qquad N \geq N_\varepsilon$$

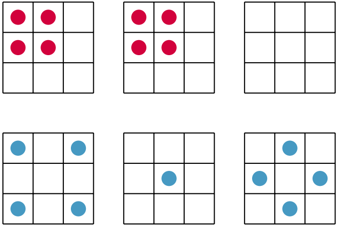
- $\|(a_0,\dots,a_{d-1})\| = 0 \quad \rightsquigarrow$ sphere

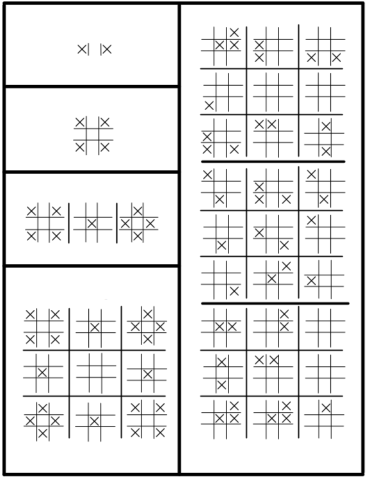
Progression-Free Sets in $\mathbb{Z}_3^n$



Lower Bounds

- $r_3(\mathbb{Z}_3^2) = 4 \quad \Rightarrow r_3(\mathbb{Z}_3^n) \gg 2^n$
- $r_3(\mathbb{Z}_3^3) = 9 \quad \Rightarrow r_3(\mathbb{Z}_3^n) \gg 2.08^n$
- $r_3(\mathbb{Z}_3^4) = 20 \quad \Rightarrow r_3(\mathbb{Z}_3^n) \gg 2.11^n$
- $r_3(\mathbb{Z}_3^5) = 45 \quad \Rightarrow r_3(\mathbb{Z}_3^n) \gg 2.14^n$
- $r_3(\mathbb{Z}_3^6) = 112 \Rightarrow r_3(\mathbb{Z}_3^n) \gg 2.19^n$
- $r_3(\mathbb{Z}_3^n) \gg 2.21^n$
  (Calderbank, Fishburn)





- Situation gets complicated very fast.
- It is difficult to find maximal progression-free sets in high dimensions.

  $\rightsquigarrow$ **bounds**

Progression-Free Sets in $\mathbb{Z}_p^n$

Theorem (Lin–Wolf 2010)

If $k \leq p$, then we have

$$r_k(\mathbb{Z}_p^n) \geq \left(p^{2(k-1)} + p^{k-1} - 1\right)^{\frac{n}{2k}} \approx p^{\frac{(k-1)n}{k}}.$$

Theorem (Elsholtz–Pach 2020)

For $p \geq 5$ and some explicitly given constant d_p, we have

$$r_3(\mathbb{Z}_p^n) \geq \frac{d_p}{\sqrt{n}} \left(\frac{p+1}{2}\right)^n.$$

Basic idea of the construction:

For vectors in the progression-free set,

select a "good" set of digits $D \subseteq \mathbb{Z}_p$

and only use these digits for the vectors.

$\hookrightarrow$ **sets of size** $(|D| - o(1))^n$

Progression-Free Sets in $\mathbb{Z}_p^n$

Theorem (Elsholtz–Klahn–L 2020+)

For $k \geq 5$ odd we have

$$r_k(\mathbb{Z}_p^n) \gg \left(\left(1 - \frac{2}{k+1}\right)p - o(1)\right)^n.$$

For $k \geq 4$ even and $p \equiv -1 \mod k$ we have

$$r_k(\mathbb{Z}_p^n) \gg \left(\left(1 - \frac{2}{k}\right)p + 1 - o(1)\right)^n.$$

(improving on $p^{(k-1)/k}$)

Theorem (Elsholtz–Klahn–L 2020+)

$r_5(\mathbb{Z}_{23}^n) \gg (17 - o(1))^n$ (improving on 12.28^n)

$r_7(\mathbb{Z}_{29}^n) \gg (24 - o(1))^n$ (improving on 17.92^n)

Overview of the Construction

For a fixed prime p and

some **set of digits** $D \subseteq \mathbb{Z}_p$,

we consider the set

$$S(D,n) := \left\{ (a_1,\dots,a_n) \in D^n \,\middle|\, \forall d \in D\colon a_i = d \text{ for } \frac{n}{|D|} \text{ values of } i \right\}.$$

We call D **good** if $S(D,n)$ is a cap for all appropriate $n \in \mathbb{N}$.
By Stirling's formula, we obtain

$$|S(D,n)| = \prod_{\ell=0}^{|D|-1} \binom{n - \frac{\ell n}{|D|}}{\frac{n}{|D|}} \sim \frac{c|D|^n}{n^\delta}$$

with

$$\delta = \frac{|D|-1}{2} \quad \text{and} \quad c = \frac{1}{\sqrt{1-\delta/|D|}} \left(\frac{|D|}{2\pi}\right)^{\delta/2}.$$

Example: $k = 3$ and $p = 11$

We choose the digit set $D = \{0, 1, 3, 4, 5\}$.

If D is good, then this implies

$$r_3(\mathbb{Z}_{11}^n) \gg \frac{5^n}{n^2}.$$

Progressions in D:

$$\{(1, 3, 5), (3, 4, 5), (5, 3, 1), (5, 4, 3)\}$$
$$\hookrightarrow \{(3, 4, 5), (5, 4, 3)\} \quad \to \emptyset$$

$\Rightarrow S(D, n)$ does not contain any arithmetic progressions

Theorem (Elsholtz–Klahn–L 2020+)

For $k \geq 5$ odd we have

$$r_k(\mathbb{Z}_p^n) \gg \left(\left(1 - \frac{2}{k+1}\right)p - o(1)\right)^n.$$

For $k \geq 4$ even and $p \equiv -1 \mod k$ we have

$$r_k(\mathbb{Z}_p^n) \gg \left(\left(1 - \frac{2}{k}\right)p + 1 - o(1)\right)^n.$$

(improving on $p^{(k-1/k)}$)

Theorem (Elsholtz–Klahn–L 2020+)

$$r_5(\mathbb{Z}_{23}^n) \gg (17 - o(1))^n \quad \text{(improving on } 12.28^n\text{)}$$

$$r_7(\mathbb{Z}_{29}^n) \gg (24 - o(1))^n \quad \text{(improving on } 17.92^n\text{)}$$

Next Objects of Interest: Caps

Definition

An affine (resp. projective) **cap** is a
subset of the affine (resp. projective) space
in which **no three points lie on a line**.

We mainly consider *affine caps* in $\mathbb{Z}_p^n = (\mathbb{Z}/p\mathbb{Z})^n$ for primes p, and we set

$$r_k(\mathbb{Z}_p^n) := \max\{|S| : S \text{ is a cap in } \mathbb{Z}_p^n\}.$$

Aim:

construction of large caps in $\mathbb{Z}_p^n$ for primes p and arbitrary dimension n

$\hookrightarrow$ **good lower bounds** for $C(\mathbb{Z}_p^n)$

Since every subset of an affine space can be embedded into the projective space, our lower bounds also hold in the projective case.

Upper Bounds

For $p \in \{3, 4, 5\}$, we have

$$\text{“no three points on a line”} \iff \text{“no three points in AP”}.$$

Theorem

- Ellenberg–Gijswijt (2016): $C(\mathbb{Z}_3^n) \leq 2.756^n$,
- Croot–Lev–Pach (2016): $C(\mathbb{Z}_4^n) \leq 3.611^n$.

Theorem (Blasiak–Church–Cohn et al. 2017)

We have

$$C(\mathbb{Z}_p^n) \leq (J(p)p)^n,$$

where

$$J(p) = \frac{1}{p} \min_{0<t<1} \frac{1 - t^p}{(1-t)t^{(p-1)/3}}.$$

Previously Known Lower Bounds

Best known general constructions so far are "**local**":

take the **tensor product of a large cap in small dimension**

For a fixed prime p, we have:

Theorem (Bose 1947)

$$C(\mathbb{Z}_p^3) = p^2 \quad \text{and so} \quad C(\mathbb{Z}_p^n) \gg p^{2n/3}.$$

Theorem (Edel–Bierbrauer 2004)

$$C(\mathbb{Z}_p^6) \geq p^4 + p^2 - 1 \quad \text{and so} \quad C(\mathbb{Z}_p^n) \gg (p^4 + p^2 - 1)^{n/6}.$$

Theorem (Elsholtz–Pach 2020)

$$C(\mathbb{Z}_4^n) \gg \frac{3^n}{\sqrt{n}} \quad \text{and} \quad C(\mathbb{Z}_5^n) \gg \frac{3^n}{\sqrt{n}}.$$

Our Results on Caps

Theorem (Elsholtz–L 2020+)

$$C(\mathbb{Z}_{11}^n) \gg \frac{5^n}{n^{1.5}}, \quad C(\mathbb{Z}_{17}^n) \gg \frac{7^n}{n^{2.5}}, \quad C(\mathbb{Z}_{23}^n) \gg \frac{9^n}{n^{3.5}},$$
$$C(\mathbb{Z}_{29}^n) \gg \frac{10^n}{n^4}, \quad C(\mathbb{Z}_{41}^n) \gg \frac{12^n}{n^5}.$$

- exponential improvements for **all primes** $p \leq 41$ **with** $p \equiv 5 \bmod 6$
- "**global**" and "**digit-based**" construction based on the method of Elsholtz and Pach for progression-free sets

Comparison of the Lower Bounds

In order to get rid of the dimension in $C(\mathbb{Z}_p^n)$, we define

$$c(p) := \lim_{n\to\infty} \left(C(\mathbb{Z}_p^n)\right)^{1/n}.$$

It is known that the limit exists and $c(p) \in [2, p)$.

p	$p^{2/3}$	$(p^4 + p^2 - 1)^{1/6}$	new	improvement
5	2.92401 . . .	2.94243 . . .	**3**	1.9562%
7	3.65930 . . .	3.67139 . . .	3	
11	4.94608 . . .	4.95282 . . .	**5**	0.9526%
13	5.52877 . . .	5.53418 . . .	4	
17	6.61148 . . .	6.61528 . . .	**7**	5.8156%
19	7.12036 . . .	7.12364 . . .	6	
23	8.08757 . . .	8.09012 . . .	**9**	11.2468%
29	9.43913 . . .	9.44099 . . .	$\geq$ **10**	$\geq$ 5.9210%
31	9.86827 . . .	9.86998 . . .	≥ 8	
37	11.10370 . . .	11.10505 . . .	≥ 10	
41	11.89020 . . .	11.89138 . . .	$\geq$ **12**	$\geq$ 0.9134%

Connection and Difference to APs

Three-term arithmetic progressions are solutions of the equation

$$x - 2y + z = 0. \qquad (\star)$$

Three points x, y, $z \in \mathbb{Z}_p^n$ are **not collinear** if and only if

$$ax + by + cz \neq 0 \quad \text{for all } (a, b, c) \in \mathbb{Z}_p^3 \setminus \{(0, 0, 0)\}$$

with $a + b + c = 0$.

Without loss of generality, we can assume $a = 1$ and $b \notin \{-1, 0\}$.

Three points x, y, $z \in \mathbb{Z}_p^n$ are **not collinear** if and only if

$$x + by + (-b - 1)z \neq 0 \quad \text{for all } b \in \mathbb{Z}_p \setminus \{-1, 0\}. \qquad (\star\star)$$

$\hookrightarrow$ still $p - 2$ equations to consider

Idea: Apply the method for progression-free sets not only to $(\star)$, but also to the other equations $(\star\star)$ corresponding to "weighted progressions".

$\rightsquigarrow$ **much more involved**

Finding Good Digit Sets (I)

We fix $b \in \mathbb{Z}_p \setminus \{-1, 0\}$ and $D \subseteq \mathbb{Z}_p$, and set

$$P_b(D) = \left\{ (x, y, z) \in D^3 \,\middle|\, x + by + (-b-1)z = 0 \right\} \setminus \left\langle (1,1,1) \right\rangle.$$

Assume that there is some $n \in \mathbb{N}$ with $|D| \mid n$ such that there are 3 points

$$x = (x_1, \ldots, x_n)^\top,\ y = (y_1, \ldots, y_n)^\top,\ z = (z_1, \ldots, z_n)^\top \in S(D, n)$$

which satisfy $x + by + (-b-1)z = 0$.
$\rightsquigarrow$ **introduce variable χ_v** for each $v = (v_1, v_2, v_3) \in P_b(D)$ which describes the number of occurrences of v in the components of x, y, z, i.e.,

$$\chi_v = \left| \left\{ i \in \{1, \ldots, n\} \,\middle|\, (x_i, y_i, z_i) = v \right\} \right|.$$

Since every digit d in D has to occur the same number of times, we find

$$\sum_{\substack{v \in P_b(D) \\ v_1 = d}} \chi_v = \sum_{\substack{v \in P_b(D) \\ v_2 = d}} \chi_v \quad \text{and} \quad \sum_{\substack{v \in P_b(D) \\ v_1 = d}} \chi_v = \sum_{\substack{v \in P_b(D) \\ v_3 = d}} \chi_v.$$

Finding Good Digit Sets (II)

$$\sum_{\substack{v\in P_b(D)\\ v_1=d}} \chi_v = \sum_{\substack{v\in P_b(D)\\ v_2=d}} \chi_v \quad \text{and} \quad \sum_{\substack{v\in P_b(D)\\ v_1=d}} \chi_v = \sum_{\substack{v\in P_b(D)\\ v_3=d}} \chi_v \qquad (\star)$$

$S(D,n)$ does not contain x, y, z with $x+by+(-b-1)z=0$ for any appropriate n. $\iff$ System $(\star)$ has no non-trivial non-negative integral solution $\chi=(\chi_v \mid v\in P_b(D))$.

Hence, to show the "goodness" of some D, one has to ensure that

$$\mathcal{P}=\{\chi\in\mathbb{Z}^{\ell}_{\geq 0} \mid A\cdot\chi=0\}$$

is empty, where the matrix A represents $(\star)$.

$\rightsquigarrow$ **integer programming**

- Appropriate software is available. ☺
- Checking the emptiness of $\mathcal{P}$ is NP-complete. ☹

$\rightsquigarrow$ **simpler conditions required**

$$P_b(D) = \left\{ (x, y, z) \in D^3 \,\middle|\, x + by + (-b-1)z = 0 \right\} \setminus \left\langle (1,1,1) \right\rangle$$

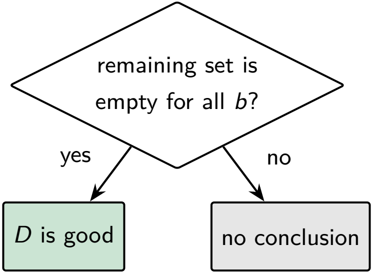
If there is some $r \in \{1, 2, 3\}$ and a digit $d \in D$ such that

d **does not occur in position** r **in any triple of** $P_b(D)$, then

remove all triples of $P_b(D)$ **which contain** d **in any position**.

Proceed recursively with the remaining set.

Else: stop.



Equivalent Equations

We have already see:

The "goodness" of (D, D') can be determined via $P_b(D)$.
The order of elements in $(x, y, z) \in P_b(D)$ does not matter.

- $(x, y, z) \in P_b(D) \iff (x, z, y) \in P_{-b-1}(D)$
  $\hookrightarrow$ **only one** of the equations

  $$x + by + (-b-1)z = 0 \quad \text{and} \quad x + (-b-1)y + bz = 0$$

  has to be considered
- $(x, y, z) \in P_b(D) \iff (z, y, x) \in P_{(-b-1)^{-1}b}(D)$
  $\hookrightarrow$ **only one** of the equations

  $$x + by + (-b-1)z = 0 \quad \text{and}$$
  $$x + (-b-1)^{-1}by + (-b-1)^{-1}z = 0$$

  has to be considered

$\rightsquigarrow$ **significant reduction** of the number of equations

Example: $p = 11$

We choose the digit set $D = \{0, 1, 3, 4, 5\}$.
If D is good, then this implies

$$C(\mathbb{Z}_{11}^n) \gg \frac{5^n}{n^2}.$$

Equivalent equations:

- $\{x - 2y + z = 0, x - 10y + 9z = 0, x - 6y + 5z = 0\}$,
- $\{x - 3y + 2z = 0, x - 7y + 6z = 0, x - 9y + 8z = 0,$
  $x - 5y + 4z = 0, x - 8y + 7z = 0, x - 4y + 3z = 0\}$.

1. $x - 2y + z = 0$:

$$P_{-2}(D) = \{(1, 3, 5), (3, 4, 5), (5, 3, 1), (5, 4, 3)\}$$
$$\hookrightarrow \{(3, 4, 5), (5, 4, 3)\} \quad \to \emptyset$$

2. $x - 3y + 2z = 0$:

$$P_{-3}(D) = \{(1, 0, 5), (1, 3, 4), (1, 4, 0), (3, 0, 4),$$
$$(3, 1, 0), (4, 1, 5), (4, 5, 0), (5, 0, 3)\} \quad \to \emptyset$$

Projective Caps

The affine space $\mathbb{Z}_p^n$ can always embedded into the projective space of the same dimension, i.e., via

$$\mathbb{Z}_p^n \hookrightarrow \mathrm{PG}(n,p), \quad (p_1,\ldots,p_n) \mapsto (1 : p_1 : \cdots : p_n).$$

$\rightsquigarrow$ bounds on affine caps also hold for projective caps

Theorem (Bose 1947, Qvist 1952)

For an odd prime power q,
the maximal size of a cap in $\mathrm{PG}(3,q)$ is q^2+1.

These maximal caps are calles **ovoids**.

Linear Codes (I)

Usually in coding theory:

Codes and Co.

- A q-ary **linear** $[n, k, d]$-**code** C is
  a k-dimensional subspace of
  the n-dimensional vector space over $\mathrm{GF}(q)$
  with minimal Hamming distance d
- A **generator matrix** G of C is
  a $k \times n$-matrix whose rows form a basis of C.
- A **check matrix** H of C is
  a $(n-k) \times k$-matrix with $cH^\top = 0$ for all $c \in C$.

Linear Codes (II)

More convenient for our purposes:

Connection to Caps (Hill 1978)

- identify a vector with its non-zero scalar mutliples
  $\rightsquigarrow$ $[n, k, d]$-code is a $(k-1)$-dimensional subspace of $\mathrm{PG}(n-1, q)$
- cap in $\mathrm{PG}(k-1, q)$ of size n
  $\hookrightarrow$ columns of $k \times n$-matrix H
- Then H is a check matrix of a $[n, n-k, d']$-code $C^{\perp}$ with $d' \geq 4$ and its dual is a $[n, k, d]$-code.
- Also the other direction works!

Good caps often lead to good codes!

Example: largest cap in $\mathrm{PG}(5, 3)$ has size 56
$\rightsquigarrow$ ternary $[56, 6, 36]$-code

Recap

- **Progression-free sets in various settings**
  - in the integers (classical results)
  - in the affine space $\mathbb{Z}_m^n$
- **Caps**
  - in the affine space
  - in the projective space
- Connection to linear codes

Thank you for your attention!