Asymptotic Analysis of q-Recursive Sequences

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DOCTORAL PROGRAM
DISCRETE MATHEMATICS
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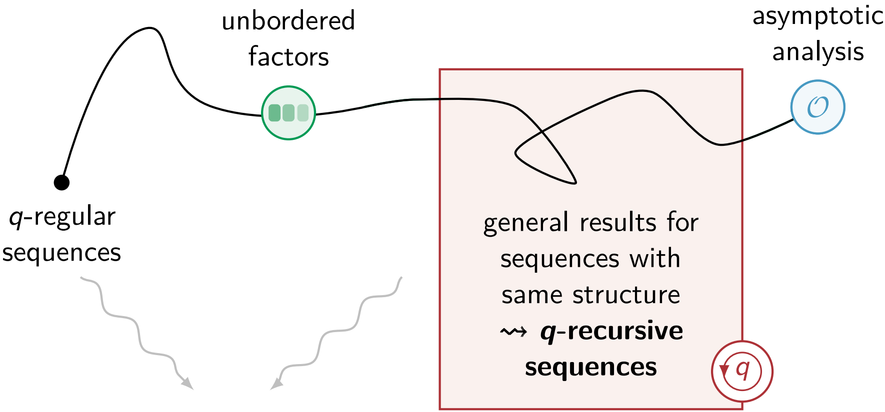
Plan for the Following 45 Minutes



q-Regular Sequences

Definition

A sequence $\boldsymbol{x}\colon \mathbb{N}_0 \to \mathbb{C}$ is **q-regular** if there are **sequences $\boldsymbol{x_1, \dots, x_D}$**
such that for every integer $i \geq 0$ and $0 \leq b < q^i$
there **exist $\boldsymbol{c_1, \dots, c_D}$** $\in \mathbb{C}$ such that $\forall n \in \mathbb{N}_0$

$$\boldsymbol{x(q^i n + b) = \sum_{j=1}^{D} c_j x_j(n)}.$$

$\rightsquigarrow$ "q-kernel" of the sequences is finitely generated

Example: Binary Sum of Digits

$s(n) =$ **sum of bits** in the binary expansion of n

$$\boldsymbol{s(2^i n + b) = s(n) + s(b)} \quad \text{for } i \geq 0 \text{ and } 0 \leq b < 2^i$$

$\Rightarrow$ s is 2-regular

Theorem (Allouche–Shallit 1992)

A sequence $\boldsymbol{x}$ is ***q*-regular** if and only if

there exist **square matrices** $A_0, \ldots, A_{q-1}$ and

a **vector-valued sequence** v with first component x such that

$$\boldsymbol{v(qn + r) = A_r v(n)}$$

holds for all $n \in \mathbb{N}_0$ and $0 \leq r \leq q - 1$.

$\rightsquigarrow (A_0, \ldots, A_{q-1}, v)$ is called ***q*-linear representation** of x

If $e_1 = (1, 0, \ldots, 0)$ and $(n)_q = d_{\ell-1} \ldots d_0$, then

$$x(n) = e_1 A_{d_{\ell-1}} \cdots A_{d_0} v(0).$$

Linear Representations – Example

$$v(qn+r) = A_r v(n)$$

Binary Sum of Digits s:
We set

$$v(n) := \begin{pmatrix} s(n) \\ 1 \end{pmatrix},$$

then

$$v(2n) = \begin{pmatrix} s(2n) \\ 1 \end{pmatrix} = \begin{pmatrix} s(n) \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{=:A_0} v(n)$$

and

$$v(2n+1) = \begin{pmatrix} s(2n+1) \\ 1 \end{pmatrix} = \begin{pmatrix} s(n)+1 \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{=:A_1} v(n).$$

$\Rightarrow$ (A_0, A_1, v) is a 2-linear representation of s.

Asymptotics for the Summatory Function

Goal: Given a q-regular sequence x, find asymptotics for

$$X(N) = \sum_{0 \le n < N} x(n).$$

- $(A_0, \ldots, A_{q-1}, v)$ linear representation of x
- $R > 0$ such that $\|A_{d_1} \cdots A_{d_\ell}\| = O(R^\ell)$ holds for all $\ell \in \mathbb{N}_0$ and $d_1, \ldots, d_\ell \in \{0, \ldots, q-1\}$ $\rightsquigarrow$ **joint spectral radius**

Theorem (Heuberger–Krenn 2018)

$$X(N) = \sum_{\substack{\lambda \in \sigma(A_0 + \cdots + A_{q-1}) \\ |\lambda| > R}} N^{\log_q \lambda} \sum_{0 \le i < m(\lambda)} (\log N)^i \cdot \Phi_{\lambda i}(\{\log_q N\}) + O\bigl(N^{\log_q R} (\log N)^{m'}\bigr)$$

- with 1-**periodic continuous functions** $\Phi_{\lambda i}$
- Fourier coefficients can be computed with arbitrary precision

Unbordered Factors in the Thue–Morse Sequence

- **Thue–Morse Sequence t:**

$$t(n) := s(n) \bmod 2$$

as an infinite word:

$$t = 01101001100\ldots$$

- **Unbordered Factors:**

Bordered Factor:	same prefix as suffix	$t = 0[110100110]0\ldots$
Unbordered Factor:	not bordered	$t = 011[010011]00\ldots$

$f(n)$ ... number of unbordered factors of length n in t

Theorem (Consequence of Goč–Mousavi–Shallit 2013)

$$\begin{aligned}
f(8n) &= \boxed{2f(4n),} & (n \geq 1)\\
f(8n+1) &= f(4n+1), & (n \geq 0)\\
f(8n+2) &= f(4n+1) + f(4n+3), & (n \geq 1)\\
f(8n+3) &= -f(4n+1) + f(4n+2), & (n \geq 2)\\
f(8n+4) &= 2f(4n+2), & (n \geq 0)\\
f(8n+5) &= f(4n+3), & (n \geq 0)\\
f(8n+6) &= -f(4n+1) + f(4n+2) + f(4n+3), & (n \geq 2)\\
f(8n+7) &= 2f(4n+1) + f(4n+3). & (n \geq 3)
\end{aligned}$$

$$\hookrightarrow \quad B_0 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad B_1 = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 2 & 0 & 1 \end{pmatrix}$$

q-Recursive Sequences

Definition

Let $M > m \geq 0$ and $\ell \leq 0 \leq u$ be integers.

A sequence $\boldsymbol{x}\colon \mathbb{N}_0 \to \mathbb{C}$ is ***q*-recursive**

with **exponents $\boldsymbol{M}$ and $\boldsymbol{m}$** and **bounds $\boldsymbol{\ell}$ and $\boldsymbol{u}$**

if for all $0 \leq r < q^M$ and $\ell \leq k \leq u$ there are $c_{r,k} \in \mathbb{C}$ with

$$x(\boldsymbol{q}^{\boldsymbol{M}} n + r) = \sum_{\ell \leq k \leq u} c_{r,k}\, x(\boldsymbol{q}^{\boldsymbol{m}} n + k)$$

for all $n \geq n_0$.

Main Result on q-Recursive Sequences

Theorem (HKL 2021)

Let $\boldsymbol{x}$ be **$\boldsymbol{q}$-recursive** with exponents M and m and bounds ℓ and u and

$$\ell' = \left(\left\lfloor \frac{\ell q^{M-m} - q^M + 1}{q^{M-m} - 1} \right\rfloor + 1 \right) [\![\ell < 0]\!] \quad \text{as well as}$$

$$u' = \max\left\{ \left\lfloor \frac{u q^{M-m} + q^M - q^m}{q^{M-m} - 1} \right\rfloor - 1,\ q^m - 1 \right\}.$$

Then $\boldsymbol{x}$ **is $\boldsymbol{q}$-regular** with linear representation
$v(n) = (v_0(n), \dots, v_{m-1}(n), v_m(n), \dots, v_{M-1}(n))^\top$, where

$$v_j(n) = \begin{pmatrix} x(q^j n) \\ \vdots \\ x(q^j n + q^j - 1) \end{pmatrix} \text{ for } 0 \le j < m \text{ and}$$

$$v_j(n) = \begin{pmatrix} x(q^j n + \ell') \\ \vdots \\ x(q^j n + q^j - q^m + u') \end{pmatrix} \text{ for } m \le j < M.$$

Idea of the Proof

- **Verify:** There exist matrices $A_0, \ldots, A_{q-1}$ with

$$v(qn + r) = A_r v(n) \quad \text{for all } 0 \leq r < q \text{ and } n \in \mathbb{N}_0.$$

$\rightsquigarrow$ each entry of $v(qn + r)$ is a linear combination of entries of $v(n)$

- Distinguish between the blocks v_j of v.
- Show that bounds ℓ' and u' are correct.

Special Case of q-Recursive Sequences

Proposition (HKL 2021)

Let $m \geq 0$ and $\boldsymbol{x}$ be 2-recursive with exponents $\boldsymbol{M = m+1}$ and $\boldsymbol{m}$ and bounds $\boldsymbol{\ell = 0}$ and $\boldsymbol{u = 2^m - 1}$.

$$\rightsquigarrow x(2^{m+1}n+r) = \sum_{k=0}^{2^m-1} c_{r,k}\, x(2^m n+k)$$

Set $\boldsymbol{u' := 2^m - 1}$ and $B_0 := (c_{r,k})_{\substack{0\leq r\leq 2^m-1\\ 0\leq k\leq 2^m-1}}$ and $B_1 := (c_{r,k})_{\substack{2^m\leq r\leq 2^{m+1}-1\\ 0\leq k\leq 2^m-1}}$.

Then $\boldsymbol{x}$ **is 2-regular** with linear representation (A_0, A_1, v), where

- $v(n) = (v_0(n), \ldots, v_{m-1}(n), v_m(n))^\top$ with

$$v_j(n) = \begin{pmatrix} x(2^j n) \\ \vdots \\ x(2^j n + 2^j - 1) \end{pmatrix} \text{ for } 0 \leq j < m \quad \text{and} \quad v_m(n) = \begin{pmatrix} x(2^m n) \\ \vdots \\ x(2^m n + u') \end{pmatrix},$$

- and

$$A_0 = \begin{pmatrix} J_{00} & J_{01} \\ 0 & \boldsymbol{B_0} \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} J_{10} & J_{11} \\ 0 & \boldsymbol{B_1} \end{pmatrix}.$$

Moreover, we have $\sigma(A_0 + A_1) = \sigma(B_0 + B_1) \cup \{0\}$.

Unbordered Factors in the Thue–Morse Sequence

$f(n) \ldots$ number of unbordered factors of length n in t

Theorem (Goč–Mousavi–Shallit 2013)

$$
\begin{aligned}
f(8n) &= 2f(4n), & (n \geq 1)\\
f(8n+1) &= f(4n+1), & (n \geq 0)\\
f(8n+2) &= f(4n+1) + f(4n+3), & (n \geq 1)\\
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f(8n+7) &= 2f(4n+1) + f(4n+3). & (n \geq 3)
\end{aligned}
$$

B_0

B_1

Corollary

The sequence f is "nearly" 2-recursive

with exponents $M = 3$ and $m = 2$ and bounds $\ell = 0$ and $u = 2^2 - 1$.

Linear Represenation of f

Proposition leads to "nearly" linear representation (A_0, A_1, v) with

$$v(n) = \begin{pmatrix} f(n) \\ f(2n) \\ f(2n+1) \\ f(4n) \\ f(4n+1) \\ f(4n+2) \\ f(4n+3) \end{pmatrix}$$

as well as

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 & 1 \end{pmatrix}.$$

$\hookrightarrow v(2n) = A_0 v(n)$ and $v(2n+1) = A_1 v(n)$ for $\boldsymbol{n \geq 3}$

Asymptotics for the Number of Unbordered Factors

Correction for $0 \le n \le 2$ yields a linear representation $(\widetilde{A}_0, \widetilde{A}_1, \widetilde{v})$ with

$$\begin{aligned}\sigma(\widetilde{A}_0 + \widetilde{A}_1) &= \sigma(A_0 + A_1) \cup \{0, 1\} \\ &= \sigma(B_0 + B_1) \cup \{0\} \cup \{0, 1\} = \{1 - \sqrt{3}, 0, 1, 2, 1 + \sqrt{3}\}.\end{aligned}$$

- joint spectral radius $\rho(\widetilde{A}_0, \widetilde{A}_1) = 2$
  $\hookrightarrow$ choose $\boldsymbol{R = 2}$
- $\kappa := \log_2(1 + \sqrt{3}) = 1.449984313\ldots$

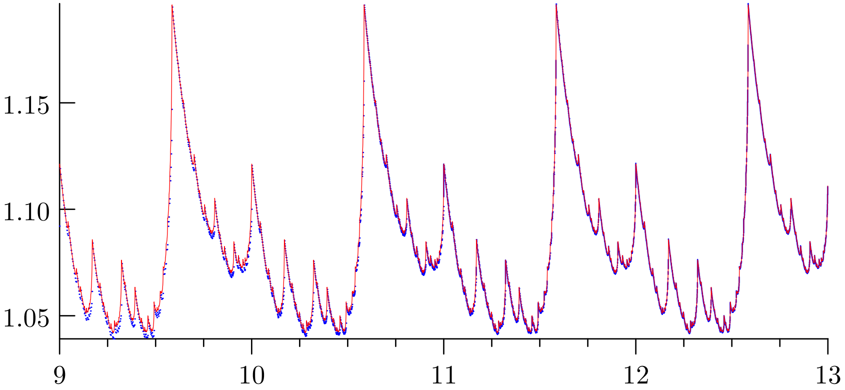
Theorem (HKL 2021)

$$F(N) = \sum_{0 \le n < N} f(n) = N^{\kappa}\Phi(\{\log_2 N\}) + O(N \log N)$$

- with a 1-**periodic function** Φ
- **Fourier coefficients** of Φ can be computed efficiently

Periodic Fluctuation in the Main Term

$$F(2^u) = 2^{u\kappa}\Phi(u) + O(2^u u)$$



— $F(2^u)/2^{u\kappa}$

— $\Phi(u)$ approximated by its trigonometric polynomial of degree 2000

Further Results

- **Linear representation and asymptotics** for
  - Stern's diatomic sequence
  - the number of non-zero elements in a generalized Pascal's triangle
- Some insights in the **growth of matrix products**
- Functional equation for the corresponding **Dirichlet series**
- **Implementation in `SageMath`** (Version 9.4)
  - `Input:` recurrence relations of a q-recursive sequence
  - `Output:`
    - linear representation
    - asymptotics incl. Fourier coefficients (combined with an implementation of Heuberger/Krenn)